Climbing Scalars, Kasner Solutions and the Big-Bang Singularity

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Summary

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Consider a theory consisting of a scalar field $\phi$ minimally coupled to gravity described by the following action

$$S = \frac{1}{2} \int d^{d+1}x \sqrt{-g} \left[ R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right]$$

Exact cosmological solutions for this system can be found for the particular case of an $O(d)$ symmetric gravitational background and an exponential scalar potential

$$ds^2 = -e^{2B(t)} dt^2 + e^{2A(t)} \sum_{i=1}^{d} \left(dx^i\right)^2 \quad \phi = \phi(t)$$

$$V(\phi) = 2\alpha e^{\gamma \phi}$$

It has been shown that

- For $\gamma < \gamma_c$, the scalar $\phi$ can descend or climb the potential after the Big Bang.
- For $\gamma \geq \gamma_c$, the scalar $\phi$ is forced to climb immediately after the Big Bang.
Exact Solution

It is instructive to look at the simple exact solution for the critical case $\gamma = \gamma_c$:

$$
\dot{\phi} \sim \frac{1}{2(\tau - \tau_0)} - \frac{1}{2} (\tau - \tau_0) \quad \phi \sim \phi_0 + \frac{1}{2} \left[ \log |\tau - \tau_0| - \frac{1}{2} (\tau - \tau_0)^2 \right]
$$

$$
\dot{A} \sim \frac{1}{2(\tau - \tau_0)} + \frac{1}{2} (\tau - \tau_0) \quad A \sim A_0 + \frac{1}{2} \left[ \log |\tau - \tau_0| + \frac{1}{2} (\tau - \tau_0)^2 \right]
$$

Comments

- The solution is written in terms of the parametric time $\tau$ related to the cosmological time by

$$
\frac{dt}{\sqrt{V}} \sim d\tau
$$

- The constant $\tau_0$ defines the Big Bang. Two more integration constants $\phi_0$ and $A_0$ arise, which determine the values of $\phi$ and $A$ at some later reference time.

- The scalar field $\phi$ can only emerge from the Big Bang while climbing up the potential, since the sign of its speed at early times is fully determined (early speed $- - > \tau_0$).

- The motion will revert at a time $\tau^*$ given by $\tau^* - \tau_0 = 1$, after which $\phi$ will begin to climb down the potential.

- The large-$\tau$ behavior of these solutions for $\gamma < \gamma_c$ correspond to the attractor solution [Lucchin and Matarrese, 1985].
Exponential potentials for the dilaton arise in String Theory in models with Brane Supersymmetry Breaking.

[Sugimoto 1999], [Antoniadis, Dudas, Sagnotti 1999]
[Angelantonj 1999], [Aldazabal, Uranga 1999]

The climbing solutions are perturbative near the big-bang, $g_s = e^\Phi \to 0$ for $\tau \to 0$.

Find specific signatures in Cosmology from String Theory models: a climbing phase in the early Universe before inflation could have left some imprints in CMBR.

[Dudas, Kitazawa, Patil, Sagnotti 2012]
[Sagnotti 2013]

Can help to solve the over-shooting problem (moduli running to infinity) in String Theory. Moduli can be trapped in minima because of climbing.

The KKLT toy model for stabilization of moduli in String Theory also contains a climbing scalar for certain choices of uplift potential: the axion “freezes”.

Climbing is an asymptotic phenomenon, thus allowing for approximate treatments (in the vicinity of the Big Bang) of the equations of motion applicable in more complicated scenarios.
Let us reconsider the climbing scalar example from the point of view of a local analysis in the vicinity of the Big Bang

\[ S = \frac{1}{2} \int d^{d+1}x \sqrt{-g} \left[ R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right] \]

We work with the cosmological time corresponding to the gauge choice \( B = 0 \), so that the metric has the following form

\[ ds^2 = -dt^2 + e^{2A(t)} \sum_{i=1}^{d} (dx^i)^2 \]

The equations of motion for the action above in the particular gravitational background considered are

\[
\begin{aligned}
\frac{d(d-1)}{2} \dot{A}^2 - \frac{1}{4} \dot{\phi}^2 &= \frac{1}{2} V \\
\frac{d(d-1)}{2} \dot{A}^2 + (d-1) \ddot{A} + \frac{1}{4} \dot{\phi}^2 &= \frac{1}{2} V \\
\ddot{\phi} + d \dot{A} \dot{\phi} &= -\frac{\partial V}{\partial \phi}
\end{aligned}
\]
Negligible Potential

Suppose that for $t \to t_0$ one can neglect the potential terms $V(\phi)$ and $V'(\phi)$ in the equations of motion.

Solving the resulting free equations one is lead to

$$A(t) = \frac{1}{d} \ln(t - t_0) + A_0 \quad \phi(t) = \pm \sqrt{\frac{2(d - 1)}{d}} \ln(t - t_0) + \phi_0$$

Given a scalar potential $V(\phi)$ that is a monotonically increasing function of $\phi$ (as it is the case for $V = 2\alpha e^{\gamma \phi}$ with $\alpha$ and $\gamma$ positive constants), then the solution with

$\dot{\phi} > 0$ describes a climbing scalar

$\dot{\phi} < 0$ describes a descending scalar

The free solutions above are asymptotic solutions of the system if and only if they satisfy the following constraints

$$V(\phi(t)), V'(\phi(t)) \sim \mathcal{O}(t^{-2+\epsilon})$$

such that they are indeed negligible with respect to $\ddot{A}, \dot{A}^2, \ddot{\phi}, \dot{\phi}^2 \sim \mathcal{O}(t^{-2})$. 
Exponential Potential

Plugging in the solutions we found for the scalar field yields the following asymptotic behavior of the exponential potential

\[ V = 2\alpha e^{\gamma \phi_0 t} \pm \gamma \sqrt{\frac{2(d-1)}{d}} \]

The climbing solution exists for any \( \gamma \), since in this case the exponent of \( t \) is always positive and hence larger than \(-2\).

The descending solution exists if and only if the positive constant \( \gamma \) satisfies the following inequality

\[ \gamma < \gamma_c \equiv \sqrt{\frac{2d}{d-1}} \]

Conclusion

If \( \gamma < \gamma_c \) then there exist two (asymptotic) solutions for the scalar field, one climbing and one descending.

If \( \gamma \geq \gamma_c \) then there exists only the climbing solution for the scalar field.
Introduction

Asymptotic Analysis

Non-negligible Potential

- We consider solutions where the kinetic and potential contributions are similar close to the Big Bang, and let us assume the following behavior

\[ V(\phi(t)) \sim t^{-2} \]

Exact Solution

- Restricting to the case of an exponential potential one obtains an exact solution of the following form

\[
A(t) = \frac{1}{d} \frac{\gamma_c^2}{\gamma^2} \ln t \\
\phi(t) = -\frac{2}{\gamma} \ln t + \frac{1}{\gamma} \ln \frac{1}{\alpha \gamma^2} \left( \frac{\gamma_c^2}{\gamma^2} - 1 \right)
\]

Notice that this solution describes a descending scalar with fixed “initial” condition \( \phi_0 \).

- It exists if and only if \( \gamma < \gamma_c \).
- It corresponds to the Lucchin-Matarrese attractor solution.
Climbing for \( n \) Scalar Fields

**n Scalar Fields with an Exponential Potential**

The equations of motion describing \( n \) scalar fields minimally coupled to the gravitational background depending only on the scale factor \( A(t) \) are

\[
\frac{d(d-1)}{2} \dot{A}^2 = \frac{1}{4} \sum_i \dot{\phi}_i^2 + \frac{1}{2} V(\phi)
\]

\[
\frac{d(d-1)}{2} \dot{A}^2 + (d-1) \ddot{A} = -\frac{1}{4} \sum_i \dot{\phi}_i^2 + \frac{1}{2} V(\phi)
\]

\[
\ddot{\phi}_i + d \dot{A} \dot{\phi}_i = -\frac{\partial V(\phi)}{\partial \phi_i}
\]

We consider a scalar potential consisting of a sum of exponentials

\[
V(\phi) = 2 \sum_{i=1}^{n} \alpha_i e^{\sum_{j=1}^{n} \gamma_{ij} \phi_j}
\]

with positive constants \( \gamma_{ij} \) and \( \alpha_i \) (such that \( V \) it is monotonically increasing with respect to every \( \phi_i \)).
Climbing for $n$ Scalar Fields

Negligible Potential

- The asymptotic solution for an asymptotically negligible potential is given by

$$A(t) = \frac{1}{d} \ln t + A_0 \quad \phi_i(t) = p_i \ln t + \phi_{0i}$$

- The solutions above are subject to the following constraints (Kasner–like sphere)

$$\sum_{i=1}^{n} p_i^2 = \frac{2(d-1)}{d} \quad \sum_{j=1}^{n} \gamma_{ij} p_j > -2$$

Definition of Climbing

- We ask that at least one scalar is forced to climb the potential when it emerges from the Big Bang. This is equivalent to asking that there is no solution where all the velocities $p_i$’s are negative.

- A sufficient (though not necessary) condition to have climbing (as defined above) is that

$$\exists \ i \in \{1, \ldots, n\} \text{ fixed, such that } \gamma_{i1}, \ldots, \gamma_{in} \geq \gamma_c$$

- The condition above is equivalent to demanding that we have $F_i \equiv \sum_{j=1}^{n} \gamma_{ij} p_j \leq -2$
Climbing for \( n \) Scalar Fields

**Exact Solution**

Let us impose the following constraints on the constants \( p_i \) of the scalar fields

\[
\sum_{j=1}^{n} \gamma_{ij} p_j = -2 \quad \text{such that} \quad V(\phi(t)) = 2 \sum_{i=1}^{n} \tilde{\alpha}_i t^{-2}
\]

where \( \tilde{\alpha}_i = \alpha_i e^{\sum_{j=1}^{n} \gamma_{ij} \phi_0} \)

Assuming that \( \det \gamma \neq 0 \) one finds the following exact solutions of the system with \( n \) scalars

\[
a = \frac{1}{a} \gamma_c^2 \sum_{i,j,k=1}^{n} (\gamma^{-1})_{ij} (\gamma^{-1})_{ik}
\]

\[
p_i = -2 \sum_{j=1}^{n} (\gamma^{-1})_{ij}
\]

\[
\tilde{\alpha}_i = \left[ \gamma_c^2 \sum_{j,k,l=1}^{n} (\gamma^{-1})_{jk} (\gamma^{-1})_{jl} - 1 \right] \sum_{j,k=1}^{n} (\gamma^{-1})_{kj} (\gamma^{-1})_{ki},
\]

Notice that all the “initial” conditions \( \phi_{0i} \) are determined by the equations for \( \tilde{\alpha}_i \).

These solutions are generalizations of the Lucchin-Matarrese attractor for the case of \( n \) scalar fields.
Climbing for $n$ Scalar Fields

Constraints

The exact solutions with non-negligible potential are subject to the following constraints

$$\gamma_c^2 \sum_{j,k,l=1}^{n} (\gamma^{-1})_{jk}(\gamma^{-1})_{jl} > 1$$

$$x_i \equiv \sum_{j,k=1}^{n} (\gamma^{-1})_{kj}(\gamma^{-1})_{ki} > 0 \quad \forall i$$

The conditions above arise from demanding that we have $\tilde{\alpha}_i > 0$ for all $i \in \{1, \ldots, n\}$.

We expect that these solutions describe $n$ descending scalars. It is then sufficient to consider models where at least one of the conditions above is violated (together with the condition found for negligible potential) in order to obtain climbing.
Climbing for $n$ Scalar Fields

A Two Field Example

**Negligible Potential**

$$V(\phi_1, \phi_2) = 2\alpha_1 e^{\gamma_{11}\phi_1 + \gamma_{12}\phi_2} + 2\alpha_2 e^{\gamma_{21}\phi_1 + \gamma_{22}\phi_2}$$

The solution is

$$A = \frac{1}{d} \ln t + A_0 \quad \phi_1 = p_1 \ln t + \phi_{01} \quad \phi_2 = p_2 \ln t + \phi_{02}$$

$$p_1^2 + p_2^2 = \frac{2(d - 1)}{d} \quad \rightarrow \quad \text{parametrize by angle } \theta$$

**General Climbing Condition**

One has a climbing scalar if and only if one of the following conditions is satisfied

(a) $\gamma_{11}, \gamma_{12} \geq \gamma_c$

(b) $\gamma_{21}, \gamma_{22} \geq \gamma_c$

(c) $\gamma_{11}, \gamma_{22} \geq \gamma_c$ and $\theta_1^* \geq \theta_2^*$

(d) $\gamma_{12}, \gamma_{21} \geq \gamma_c$ and $\tilde{\theta}_2^* \geq \tilde{\theta}_1^*$,

where the angles $\theta_1^*, \theta_2^*$ solve the equations

$$F_i(\theta_i^*) \equiv \sqrt{\frac{2(d - 1)}{d}} [\gamma_{i1} \cos \theta_i^* + \gamma_{i2} \sin \theta_i^*] = -2$$
Climbing for $n$ Scalar Fields
A Two Field Example

Exact Solution with $V \sim t^{-2}$

\[ A = a \ln t + A_0 \quad \phi_1 = p_1 \ln t + \phi_{01} \quad \phi_2 = p_2 \ln t + \phi_{02} \]

\[ p_1 = \frac{2}{\det \gamma} (\gamma_{12} - \gamma_{22}) \quad p_2 = \frac{2}{\det \gamma} (\gamma_{21} - \gamma_{11}) \]

\[ a = \frac{1}{d (\det \gamma)^2} \left[ (\gamma_{11} - \gamma_{21})^2 + (\gamma_{22} - \gamma_{12})^2 \right] \]

Introducing the quantities $x_1$, $x_2$ which in terms of $\gamma_{ij}$ read

\[ x_1 = \frac{1}{(\det \gamma)^2} (\gamma_{22}^2 + \gamma_{21}^2 - \gamma_{12} \gamma_{22} - \gamma_{11} \gamma_{21}) \]

\[ x_2 = \frac{1}{(\det \gamma)^2} (\gamma_{11}^2 + \gamma_{12}^2 - \gamma_{12} \gamma_{22} - \gamma_{11} \gamma_{21}) \]

The integration constants $\phi_{0i}$ are determined by the equations

\[ \tilde{\alpha}_i = \left[ \gamma_c^2 (x_1 + x_2) - 1 \right] x_i \quad i = 1, 2 \]

One can show that the inequalities $x_i > 0$ imply that both scalars have to be descending for this type of solutions, i.e. we have $p_1, p_2 < 0$. 
Let us consider some higher–derivative corrections to the system with one climbing scalar described by the action

\[ S_0 = \frac{1}{2} \int d^{d+1}x \sqrt{-g} \left[ R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right] \]

For \( \gamma \geq \gamma_c \), there was only one asymptotic solution describing a climbing scalar

\[ A = a \ln t \quad , \quad \phi = p \ln t \quad \text{with} \quad a = \frac{1}{d} \quad , \quad p = \sqrt{\frac{2(d-1)}{d}}, \]

We include corrections of order \( t^{-4+qp} \) to the action above

\[ S_1 = \frac{1}{2} \int d^{d+1}x \sqrt{-g} e^{-q\phi} \left[ \alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right] \]

\[ S_2 = \frac{1}{2} \int d^{d+1}x \sqrt{-g} e^{-q\phi} \left[ \beta_1 R(\partial \phi)^2 + \beta_2 R_{\mu\nu} \partial^\mu \phi \partial^\nu \phi + \beta_3 (\partial \phi)^4 + \beta_4 (\square \phi)^2 + \beta_5 (\partial \phi)^2 \square \phi + \beta_6 R \square \phi + \beta_7 R^{\mu\nu} \partial_\mu \partial_\nu \phi \right] \]

Notice that the degree of the singularity depends explicitly on the parameter \( q \). In String Theory, its value is determined by the string perturbation order which can generate these terms in the effective action.
Higher Derivative Corrections

- We impose two requirements on the actions $S_1$ and $S_2$
  - The equations of motion should contain only up to two derivatives.
    \[
    S_1 + S_2 = \frac{1}{2} \int d^{d+1}x \sqrt{-g} e^{-q\phi} \left[ \eta \left( R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) - 2\beta_1 G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \beta_3 (\nabla \phi)^4 + \beta_5 (\nabla \phi)^2 \Box \phi \right].
    \]
  - The climbing scalar solution should remain a solution of the equations of motion.
  - For the simple case of $q = 0$ corresponding to a correction of order $t^{-2}$ one obtains the following action that preserves the climbing solution
    \[
    S_1 + S_2 = \frac{1}{2} \int d^{d+1}x \sqrt{-g} \eta \left\{ R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{(d-2)(d-3)}{4d(d-1)} \left[ (\nabla \phi)^4 - 2\sqrt{\frac{2(d-1)}{d}} (\nabla \phi)^2 \Box \phi \right] \right\}
    \]
- Weak point of the effective field theory analysis: it is very hard to argue for the existence or absence of the descending solutions by including higher–order and higher derivative corrections to the effective action, since they are non–perturbative from the viewpoint of String Theory.
KKLT is a four–dimensional toy model for moduli stabilization in String Theory described by the following Kähler metric and superpotential

\[ K = -3 \ln(T + \bar{T}) \quad W = W_0 + ce^{-bT} \]

In the asymptotic limit we neglect the non–perturbative part \( e^{-bT} \) thus leaving us with \( V_F = 0 \). In addition there is an uplift potential of the form

\[ V_{up} = \frac{D}{(T + \bar{T})^{n_T}} \]

This is a particular case of the general one that we studied. With the standard parametrization for \( T \) the asymptotic solution is

\[ T = e^{\frac{\Phi_T}{\sqrt{3}}} + i \frac{\theta_T}{\sqrt{3}} \quad \Phi_T = \pm \frac{2}{\sqrt{3}} \ln(t) \quad \theta_T = \text{const} \]

The condition to have a climbing scalar in the KKLT scenario is the following

\[ n_T \geq 3 \]

In KKLT one has \( n = 3 \) for an F-term uplift. The case \( n = 2 \) that does not have climbing corresponds to anti–branes.
String Examples
S modulus in KKLT

- After stabilization of the remaining moduli (e.g. three–form fluxes in Type II strings) we suppose that we are left with the following effective superpotential and Kähler metric:

\[ W = W_{\text{eff}}(S) + ce^{-bT} \]

\[ K = -\ln(S + \bar{S}) - 3\ln(T + \bar{T}) \]

- The F-term potential and uplift are given by:

\[ V_F = \frac{|W_{\text{eff}}(S) - (S + \bar{S})W'_{\text{eff}}(S)|}{(S + \bar{S})(T + \bar{T})^3} \]

\[ V_{\text{up}} = \frac{D}{(S + \bar{S})^{n_S}(T + \bar{T})^{n_T}} \]

- Introduce the standard parametrization for \( T \) and \( S \):

\[ S = e^{\Phi_S} + i\theta_S \]

\[ T = e^{\frac{\Phi_T}{\sqrt{3}}} + i\frac{\theta_T}{\sqrt{3}} \]

- Asymptotic solutions to the equations of motion are of the form:

\[ \Phi_S = p_S \ln(t) \quad \Phi_T = p_T \ln(t) \quad \theta_S = \text{const} \quad \theta_T = \text{const} \quad p_S^2 + p_T^2 = \frac{4}{3} \]

- Consider that the effective superpotential for \( S \) is polynomial of degree \( n \):

\[ W_{\text{eff}}(S) = W_0 + W_1 S + \ldots + W_n S^n \]

- The condition to have a climbing \( \Phi_T \) scalar (irrespective of what \( \Phi_S \) does) is the following:

\[ n \geq 2 \quad n_S \geq \sqrt{3} \quad n_T \geq 3 \]
Conclusions

- The case of a real scalar field coupled to gravity with an exponential potential can be solved exactly (for fields depending only on time). For $\gamma \geq \gamma_c$, the scalar is forced to emerge the Big Bang while climbing up the potential. An asymptotic analysis is sufficient to decide whether or not a climbing scalar exists.

- The existence of a climbing phase before slow-roll could have in principle effects on the CMBR.

- We have considered climbing in a more general setting involving multiple scalar fields and we derived conditions for the existence of a climbing scalar. We have found exact solutions of the Lucchin-Matarrese type for systems with $n$ scalar fields.

- We have considered higher derivative corrections to the original action with a climbing scalar. Imposing that the equations of motion are still of second order and that the climbing solution is preserved we have shown that there is a unique action (up to normalization) compatible with these requirements (at order $t^{-(4+qp)}$). We have also considered other examples with higher derivatives, e.g. a DBI action.

- In the KKLT model there is again a climbing scalar. The inclusion of $S$ can spoil the climbing if the effective superpotential is linear in $S$. 