On $D = 4$ Stationary Black Holes

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Based on:


Outline

1. Black Holes in Extended $D = 4$ (Symmetric) Supergravity
2. The Global Symmetry in $D = 3$ and Orbits
3. Singular Limits to Regular Extremal Solutions
4. The Issue of Nilpotent Orbits and an Example
5. Conclusions
### Stationary, asymptotically Flat Black Holes in D=4 SUGRAS

**Supergravity bosonic field content**

- $n_S$ scalar fields $\phi^r$
- $n_V$ vector fields $A^\Lambda_\mu$
- Graviton $g_{\mu\nu}$
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- Metric: $ds^2 = -e^{2U} \left( dt + B_i^0 \, dx^i \right)^2 + e^{-2U} \, g_{ij} \, dx^i \, dx^j$
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\[
e^{-1} L = \frac{R_3}{2} - \frac{1}{2} G_{IJ}(\phi) \partial_i \phi^I \partial^i \phi^J \Rightarrow (\phi^I) \in \mathcal{M}^{(3)}_{\text{scal}} = \frac{G}{H}
\]

\( \mathcal{M}^{(3)}_{\text{scal}} = \frac{G}{H} \) is pseudo-Riemannian (indefinite signature), \( H \subset G \) is non-compact semisimple.
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- Spherical symmetry: $\phi^I = \phi^I(\tau)$, solution is a geodesic on $\mathcal{M}^{(3)}_{\text{scal}}$
Geodesic uniquely defined by initial point \( \phi^I_0 = \phi^I(\tau = 0) \) and initial velocity \( Q \in T_{\phi_0}(G/H) \)
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Isometry group \( G \) is the global symmetry of the \( D = 3 \) theory

Action of \( G \) on a geodesic \((\phi_0, Q)\)

Fix \( \phi_0 \equiv 0 \), \( G \)-orbit of geodesic is \( H \)-orbit of \( Q \in T_0 \)
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Fix $\phi_0 \equiv 0$, $G$-orbit of geodesic is $H$-orbit of $Q \in T_0$
Axisymmetric solutions $\phi = \phi(\tau, \theta)$, still defined by unique point $\phi_0 = \lim_{\tau \to 0} \phi(\tau, \theta)$ and a vector $Q \in T_{\phi_0}$. Fix $G/H$ by setting $\phi_0 = 0$, $T_{\phi_0} = T_0$. 
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Tangent space at the origin \( T_0 \sim \mathbb{R} \) subspace of the Lie algebra \( \mathfrak{g} \) of \( G \), complement of the Lie algebra \( \mathfrak{h} \) of \( H \). Space has indefinite metric. \( Q \) is a matrix in \( \mathbb{R} \).
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Velocity vector $Q$ is the Noether-charge matrix:

$$Q = \frac{1}{4\pi} \int_{S_2} * J = M_{ADM} K_0 + \Sigma^r K_r + \Sigma^{\text{NUT}} K_\bullet + p^\Lambda K_\Lambda + q_\Lambda K_{\Lambda} \in \mathfrak{R},$$

$J = J_i \, dx^i$ being the Noether current. $Q$ does not contain the angular momentum $\mathcal{J}$!
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Tangent space at the origin $T_0 \sim \mathfrak{k}$ subspace of the Lie algebra $\mathfrak{g}$ of $G$, complement of the Lie algebra $\mathfrak{h}$ of $H$. Space has indefinite metric. $Q$ is a matrix in $\mathfrak{k}$

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Define new $\mathfrak{g}$-matrix $Q_\psi$ capturing rotation:

$$Q_\psi = -\frac{3}{4\pi} \int_{S^2} \psi_{[i} J_{j]} \, dx^i \wedge dx^j = \mathcal{J} K_\bullet + \cdots \in \mathfrak{k},$$

$\psi = \partial_\varphi$ and $Q, Q_\psi$ represent independent vectors in $T_0$ [arXiv:1210.4047 [hep-th]].

Static solution $Q_\psi = 0$
Global symmetry and regularity

- Action of $G$ on the solution $\Rightarrow$ action of $H$ on $Q, Q_\psi$:

$$Q \rightarrow Q' = h^{-1} Q h \quad Q_\psi \rightarrow Q'_\psi = h^{-1} Q_\psi h \quad (h \in H)$$
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$$\frac{k}{2} \text{Tr}(Q^2) = m^2 - \frac{p^2 + q^2}{2} \quad \text{Tr}(Q^2_\psi) = \frac{J^2}{m^2} \text{Tr}(Q^2),$$
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- Regularity condition can be written in a $G$-invariant form:
  
  $$\frac{k}{2} \text{Tr}(Q^2) \geq \frac{\text{Tr}(Q^2_{\psi})}{\text{Tr}(Q^2)}$$
  
  “$=$” for extremal ($T = 0$) solutions
Singular limits...

- Limits of non-extremal axisymmetric solution studied in specific contexts (Heterotic Sugra: Cvetic, Youm 9603147; Astefanesei et al. 0606244; Kaluza-Klein Sugra: Rasheed 9505038; Larsen 9909102)
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- Harrison generators \((\mathcal{J}_M) = (\mathcal{J}_\Lambda, \mathcal{J}^\Lambda)\) in \(\mathfrak{h}\) are in one-to-one correspondence with \((\mathcal{P}^M) = (p^\Lambda, q_\Lambda)\). MASA of \(\text{Span}(\mathcal{J}_M)\) is generated by \(\{\mathcal{J}_\ell\}, \ell = 1, \ldots, p\), is defined by the normal form of the e-m charges: \(p = \text{rank}(H/m.c.s.(H))\)
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- Act on the Kerr solution \((m_K, \mathcal{J}_K)\) by means of the Harrison transformation:

\[
h = \exp \left( m_K^{\pm 1} \beta_1 \mathcal{J}_1 + \cdots + m_K^{\pm 1} \beta_p \mathcal{J}_p \right)
\]

Resulting non-extremal solution has charges in the normal form with signs depending on “\(\pm\)”. 
Extremal under-rotating [Goldstein, Katmadas, 0812.4183; Bena at al. 0902.4526] and static: After applying $h = \exp \left( \sum \ell \, m_K^{\pm 1} \beta_\ell \mathcal{J}_\ell \right)$, send $m_K \to 0$ and $\mathcal{J}_K \to 0$, keeping $\mathcal{J}_K / m_K^2$ fixed.
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The $m_{K}, J_{K} \to 0$ limits are Inönü-Wigner contractions:

\[ Q' = h^{-1} Q h, \quad Q'_{\psi} = h^{-1} Q_{\psi} h \longrightarrow Q^{(0)}, \quad Q^{(0)}_{\psi} \text{ nilpotent} \]
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Case 1: $\text{step}(Q^{(0)}_{\psi}) = \text{step}(Q^{(0)}) - 1$ and in the regularity condition

$$\frac{k}{2} \frac{\text{Tr}(Q'^{2})}{\text{Tr}(Q^{2})} \geq \frac{\text{Tr}(Q'_{\psi}^{2})}{\text{Tr}(Q_{\psi}^{2})}$$

both sides vanish separately in the limit $Q' \to Q^{(0)}$ and $Q'_{\psi} \to Q^{(0)}_{\psi}$. 
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Case 2: $Q^{(0)}$ nilpotent, $Q_{\psi}^{(0)} = 0$. 
The Issue of Nilpotent Orbits

In the limit $m_K, J_K \to 0$, keeping $J_K/m_K$ fixed, we find BPS and non-BPS ($I_4(p, q) > 0$) rotating solutions $J \neq 0$. They are singular with $Q^{(0)}_\psi, Q^{(0)}$ nilpotent of the same degree.
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- In all these limits we find the generic representative of each of the relevant $H$-orbit (seed solution with respect to $G$).
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Classifying Extremal Under-rotating and Static Solutions: Nilpotent Orbits....

Orbits of nilpotent generators $X (Q^{(0)}, Q^{(0)}_\psi)$ in $\mathfrak{h}$ under $H$: $O_X^{(H)} = H^{-1} X H$.

Classification in steps:
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**The $F_4(4)$-model**

- $D = 4$, $N = 2$ symmetric SUGRA coupled to 6 vector multiplets
- Time-reduction to $D = 3 \rightarrow \frac{G}{H} = \frac{F_4(4)}{SL(2,\mathbb{R}) \times Sp'(6,\mathbb{R})}$ (pseudo-quaternionic)
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- Complete classification of $H = \text{SL}(2,\mathbb{R}) \times \text{Sp}'(6,\mathbb{R})$-nilp. orbits in $\mathcal{K} = (2, 14')$

[W. Chemissany, P. Giaccone, D. Ruggeri and M. T.,1203.6338]
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- Defined general procedure for constructing nilpotent $H$-orbits in $T_0$;
Conclusions

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